Int. J. Math. And Appl., 8(2)(2020), 7–13
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

# On the Zeros of Polynomials with Complex Coefficients

**P. Ramulu[1,*], G. L. Reddy[2] and C. Gangadhar[2]**

1 Department of Mathematics, M.V.S Govt. Arts and Science College (Autonomous), Mahabubnagar, Telangana, India.

2 School of Mathematics and Statistics, University of Hyderabad, Telangana, India

**Abstract:** In this paper we prove some extension of the Eneström-Kakeya theorem (Let $P(z)=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n$ such that $0<a_0 \le a_1 \le ... \le a_n$ then all the zeros of P(z) lie in $|z| \le 1$) by relaxing the hypothesis in different ways we get various other results which in term generalizes.

**MSC:** 30C10, 30C15.

**Keywords:** Zeros of polynomial, Eneström-Kakeya theorem.



## 1. Introduction

To estimate the zeros of a polynomial is a long standing classical problem [4-12]. It is an interesting area of research for engineers as well as mathematicians and many results on the same topic are available in literature. The polynomials in various forms have recently come under extensive revision because of their applications in linear control systems, signal processing, electrical networks, coding theory and several areas of physical sciences, where among others, location of zeros and stability problems arise in a natural way. Existing results in the literature also show that there is a need to find bounds for special polynomials, for example,for those having restrictions on the coefficient,there is always need for refinement of results in this subject. The well known result Eneström-Kakeya theorem [3,9] in theory of the distribution of zeros of polynomials is the following:

**Theorem** $(A_1)$**.** *Let* $P(z)=\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n$ *such that* $0<a_0 \le a_1 \le ... \le a_{n-1} \le a_n$ *then all the zeros of P(z) lie in* $|z| \le 1$.

A. Joyal , G. Labelle and Q. I. Rahman [1] obtained the following generalization, by considering the coefficients to be real, instead of being only positive.

**Theorem** $(A_2)$**.** *Let* $P(z)=\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n$ *such that* $a_0 \le a_1 \le ... \le a_{n-1} \le a_n$ *then all the zeros of P(z) lie in* $|z| \le \frac{1}{|a_n|}\{a_n - a_0 + |a_0|\}$.

In the literature some attempts have been made to extend and generalize the Eneström-Kakeya theorem. Aziz and Zargar [2] relaxed the hypothesis of Eneström-Kakeya theorem in a different way and proved the following results:

* *E-mail: ramulu.purra@gmail.com*

**Theorem** $(A_3)$. *Let* $P(z)=\sum\limits_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n$ *such that for some* $k\geq 1$, $0<a_0\leq a_1\leq ...\leq a_{n-1}\leq ka_n$ *then all the zeros of* $P(z)$ *lie in* $|z+k-1|\leq k$.

**Theorem** $(A_4)$. *Let* $P(z)=\sum\limits_{i=0}^{n} a_i z^i$ *be a polynomial of degree* $n$ *such that for some* $k\geq 1$, $a_0\leq a_1\leq ...\leq a_{n-1}\leq ka_n$ *then all the zeros of* $P(z)$ *lie in* $|z+k-1|\leq \frac{1}{|a_n|}\{ka_n-a_0+|a_0|\}$.

In this paper We want to prove the following results.

**Theorem 1.1.** *Let* $P(z)=\sum\limits_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* $n$ *with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,...,n$ *such that for some* $k\geq 1, k_1\geq 1, l\geq 1, l_1\geq 1, \delta\geq 0, \eta\geq 0, a_m\neq 0, b_m\neq 0,$

$$a_0-\delta\leq a_1\leq ...\leq a_{m-1}\leq k_1a_m\leq a_{m+1}\leq ...\leq a_{n-1}\leq ka_n \quad \text{and}$$
$$b_0-\eta\leq b_1\leq ...\leq b_{m-1}\leq l_1b_m\leq b_{m+1}\leq ...\leq b_{n-1}\leq lb_n$$

*then all the zeros of* $P(z)$ *lie in*

$$|z|\leq \frac{1}{|\alpha_n|}\big[k(a_n+|a_n|)+l(b_n+|b_n|)+|a_0|+|b_0|+2(k_1-1)|a_m|+2(l_1-1)|b_m|-(a_0+b_0+|a_n|+|b_n|)+2\delta+2\eta\big].$$

**Corollary 1.2.** *Let* $P(z)=\sum\limits_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* $n$ *with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,...,n$ *such that for some* $k\geq 1, k_1\geq 1, \delta\geq 0, \eta\geq 0, a_m\neq 0, b_m\neq 0,$

$$a_0-\delta\leq a_1\leq ...\leq a_{m-1}\leq k_1a_m\leq a_{m+1}\leq ...\leq a_{n-1}\leq ka_n \quad \text{and}$$
$$b_0-\eta\leq b_1\leq ...\leq b_{m-1}\leq k_1b_m\leq b_{m+1}\leq ...\leq b_{n-1}\leq kb_n$$

*then all the zeros of* $P(z)$ *lie in*

$$|z|\leq \frac{1}{|\alpha_n|}\big[k(a_n+b_n+|a_n|+|b_n|)+|a_0|+|b_0|+2k_1(|a_m|+|b_m|)-(a_0+b_0+|a_n|+|b_n|)+2(\delta+\eta-|a_m|-|b_m|)\big].$$

**Corollary 1.3.** *Let* $P(z)=\sum\limits_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* $n$ *with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,...,n$ *such that for some* $k\geq 1, \delta\geq 0,$

$$a_0-\delta\leq a_1\leq ...\leq a_{m-1}\leq a_m\leq a_{m+1}\leq ...\leq a_{n-1}\leq ka_n \quad \text{and}$$
$$b_0-\delta\leq b_1\leq ...\leq b_{m-1}\leq b_m\leq b_{m+1}\leq ...\leq b_{n-1}\leq kb_n$$

*then all the zeros of* $P(z)$ *lie in*

$$|z|\leq \frac{1}{|\alpha_n|}\big[k(a_n+b_n+|a_n|+|b_n|)+|a_0|+|b_0|-(a_0+b_0+|a_n|+|b_n|)+4\delta\big].$$

**Corollary 1.4.** *Let* $P(z)=\sum\limits_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* $n$ *with* $Re(\alpha_i)=a_i$ *and* $Im(\alpha_i)=b_i$ *for* $i=0,1,2,...,n$ *such that for some* $k\geq 1, k_1\geq 1, \delta\geq 0, a_m\neq 0$

$$a_0-\delta\leq a_1\leq ...\leq a_{m-1}\leq k_1a_m\leq a_{m+1}\leq ...\leq a_{n-1}\leq ka_n \quad \text{and}$$
$$b_0\leq b_1\leq ...\leq b_{m-1}\leq b_m\leq b_{m+1}\leq ...\leq b_{n-1}\leq b_n$$

*then all the zeros of* $P(z)$ *lie in*

$$|z|\leq \frac{1}{|\alpha_n|}\big[k(a_n+|a_n|)+b_n+|a_0|+|b_0|+2(k_1-1)|a_m|-(a_0+b_0+|a_n|)+2\delta\big].$$

**Corollary 1.5.** *Let $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ be a polynomial of degree $n$ with $Re(\alpha_i)=a_i$ and $Im(\alpha_i)=b_i$ for $i=0,1,2,...,n$ such that*

$$a_0\le a_1\le ...\le a_{m-1}\le a_m\le a_{m+1}\le ...\le a_{n-1}\le a_n \quad and$$
$$b_0\le b_1\le ...\le b_{m-1}\le b_m\le b_{m+1}\le ...\le b_{n-1}\le b_n$$

*then all the zeros of P(z) lie in*

$$|z|\le \frac{1}{|\alpha_n|}\big[a_n+b_n+|a_0|+|b_0|-(a_0+b_0)\big].$$

**Corollary 1.6.** *Let $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ be a polynomial of degree $n$ with $Re(\alpha_i)=a_i>0$ and $Im(\alpha_i)=b_i>0$ for $i=0,1,2,...,n$ such that for some $k\ge 1,k_1\ge 1,\delta\ge 0,\eta\ge 0,a_m\ne 0,b_m\ne 0$,*

$$a_0-\delta\le a_1\le ...\le a_{m-1}\le k_1a_m\le a_{m+1}\le ...\le a_{n-1}\le ka_n \quad and$$
$$b_0-\eta\le b_1\le ...\le b_{m-1}\le k_1b_m\le b_{m+1}\le ...\le b_{n-1}\le kb_n$$

*then all the zeros of P(z) lie in*

$$|z|\le \frac{1}{|\alpha_n|}\big[(2k-1)(a_n+b_n)+(2k_1-1)(a_m+b_m)+2\delta+2\eta\big].$$

**Remark 1.7.**

*(1). By taking $b_i=0$ in Corollary 4, then it reduces to Theorem $A_2$.*

*(2). By taking $b_i=0$ and $a_i>0$ for $i=0,1,2,...,n$ in Corollary 4, then it reduces to Theorem $A_1$.*

*(3). By taking $k=l,k_1=l_1$ in Theorem 1, then it is reduces to Corollary 1.*

*(4). By taking $l=k,l_1=k_1=1$ and $\delta=\eta$ in Theorem 1, then it is reduces to Corollary 2.*

*(5). By taking $l=l_1=1$ and $\eta=0$in Theorem 1, then it is reduces to Corollary 3.*

*(6). By taking $l=k_1=l_1=k=1$ and $\delta=\eta=0$ in Theorem 1, then it is reduces to Corollary 4.*

*(7). By taking $b_i>0$ and $a_i>0$ for $i=0,1,2,...,n$ in Corollary 1, then it reduces to Corollary 5.*

**Theorem 1.8.** *Let $P(z)=\sum_{i=0}^{n}\alpha_i z^i$ be a polynomial of degree $n$ with $Re(\alpha_i)=a_i$ and $Im(\alpha_i)=b_i$ for $i=0,1,2,...,n$ such that for some $0<r\le 1,0<s\le 1,0<x\le 1,0<y\le 1,\delta\ge 0,\eta\ge 0,a_m\ne 0,b_m\ne 0$,*

$$ra_n\le a_{n-1}\le ...\le a_{m+1}\le sa_m\le a_{m-1}\le ...\le a_1\le a_0+\delta \quad and$$
$$xb_n\le b_{n-1}\le ...\le b_{m+1}\le yb_m\le b_{m-1}\le ...\le b_1\le b_0+\eta$$

*then all the zeros of P(z) lie in*

$$|z|\le \frac{1}{|\alpha_n|}\Big[a_0+b_0+|a_0|+|b_0|+|a_n|+|b_n|-r(a_n+|a_n|)-x(b_n+|b_n|)+2\big[|a_m|+|b_m|-s|a_m|-y|b_m|+\delta+\eta\big]\Big].$$

**Corollary 1.9.** *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* $n$ *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0<r \leq 1, 0<s \leq 1, \delta \geq 0, \eta \geq 0, a_m \neq 0, b_m \neq 0,$

$$ra_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq sa_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0 + \delta \quad \text{and}$$

$$rb_n \leq b_{n-1} \leq ... \leq b_{m+1} \leq sb_m \leq b_{m-1} \leq ... \leq b_1 \leq b_0 + \eta$$

*then all the zeros of* $P(z)$ *lie in*

$$|z| \leq \frac{1}{|\alpha_n|}\Big[a_0 + b_0 + |a_0| + |b_0| + |a_n| + |b_n| - r(a_n + |a_n| + b_n + |b_n|) + 2\big[|a_m| + |b_m| - s(|a_m| + |b_m|) + \delta + \eta\big]\Big].$$

**Corollary 1.10.** *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* $n$ *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0<r \leq 1, \delta \geq 0,$

$$ra_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq a_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0 + \delta \quad \text{and}$$

$$rb_n \leq b_{n-1} \leq ... \leq b_{m+1} \leq b_m \leq b_{m-1} \leq ... \leq b_1 \leq b_0 + \delta$$

*then all the zeros of* $P(z)$ *lie in*

$$|z| \leq \frac{1}{|\alpha_n|}\Big[a_0 + b_0 + |a_0| + |b_0| + |a_n| + |b_n| - r(a_n + b_n + |a_n| + |b_n|) + 4\delta\Big].$$

**Corollary 1.11.** *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* $n$ *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0<r \leq 1, 0<s \leq 1, \delta \geq 0, a_m \neq 0,$

$$ra_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq sa_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0 + \delta \quad \text{and}$$

$$b_n \leq b_{n-1} \leq ... \leq b_{m+1} \leq b_m \leq b_{m-1} \leq ... \leq b_1 \leq b_0$$

*then all the zeros of* $P(z)$ *lie in*

$$|z| \leq \frac{1}{|\alpha_n|}\Big[a_0 + b_0 + |a_0| + |b_0| + |a_n| + |b_n| - r(a_n + |a_n|) - b_n + 2\big[|a_m| - s|a_m| + \delta\big]\Big].$$

**Corollary 1.12.** *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* $n$ *with* $Re(\alpha_i) = a_i$ *and* $Im(\alpha_i) = b_i$ *for* $i = 0, 1, 2, ..., n$ *such that*

$$a_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq a_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0 \quad \text{and}$$

$$b_n \leq b_{n-1} \leq ... \leq b_{m+1} \leq b_m \leq b_{m-1} \leq ... \leq b_1 \leq b_0$$

*then all the zeros of* $P(z)$ *lie in*

$$|z| \leq \frac{1}{|\alpha_n|}\Big[a_0 + b_0 + |a_0| + |b_0| - a_n - b_n\Big].$$

**Corollary 1.13.** *Let* $P(z) = \sum_{i=0}^{n} \alpha_i z^i$ *be a polynomial of degree* $n$ *with* $Re(\alpha_i) = a_i>0$ *and* $Im(\alpha_i) = b_i>0$ *for* $i = 0, 1, 2, ..., n$ *such that for some* $0<r \leq 1, 0<s \leq 1, 0<x \leq 1, 0<y \leq 1, \delta \geq 0, \eta \geq 0, a_m \neq 0, b_m \neq 0,$

$$ra_n \leq a_{n-1} \leq ... \leq a_{m+1} \leq sa_m \leq a_{m-1} \leq ... \leq a_1 \leq a_0 + \delta \quad \text{and}$$

$$xb_n \le b_{n-1} \le ... \le b_{m+1} \le yb_m \le b_{m-1} \le ... \le b_1 \le b_0 + \eta$$

*then all the zeros of P(z) lie in*

$$|z| \le \frac{1}{|\alpha_n|}\Big[a_0 + b_0 + |a_0| + |b_0| + (1-2r)a_n + (1-2x)b_n + 2\big[(1-s)a_m + (1-y)b_m + \delta + \eta\big]\Big].$$

**Remark 1.14.**

*(1). By taking $r = x, s = y$ in Theorem 2, then it is reduces to Corollary 6.*

*(2). By taking $x = r, s = y = 1$ and $\delta = \eta$ in Theorem 2, then it is reduces to Corollary 7.*

*(3). By taking $x = y = 1$ and $\eta = 0$in Theorem 2, then it is reduces to Corollary 8.*

*(4). By taking $r = s = x = y = 1$ and $\delta = \eta = 0$ in Theorem 2, then it is reduces to Corollary 9.*

*(5). By taking $b_i$>0 and $a_i$>0 for $i = 0, 1, 2, ..., n$ in Theorem 2, then it reduces to Corollary 10.*

## 2. Proof of the Theorems

*Proof of Theorem 1.1.* Let $P(z) = \alpha_n z^n + \alpha_{n-1}z^{n-1} + ... + \alpha_{m-1}z^{m-1} + \alpha_m z^m + \alpha_{m+1}z^{m+1} + ... + \alpha_1 z + \alpha_0$ be a polynomial of degree n. Then consider the polynomial $Q(z) = (1-z)P(z)$ so that

$$\begin{aligned}Q(z) =& -\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + ... + (\alpha_{m+1} - \alpha_m)z^{m+1} + (\alpha_m - \alpha_{m-1})z^m + ... + (\alpha_1 - \alpha_0)z + \alpha_0.\\ =& -\alpha_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{m+1} - a_m)z^{m+1} + (a_m - a_{m-1})z^m + ... + (a_1 - a_0)z + a_0 +\\ &+ i\big\{(b_n - b_{n-1})z^n + ... + (b_{m+1} - b_m)z^{m+1} + (b_m - b_{m-1})z^m + ... + (b_1 - b_0)z + b_0\big\}.\end{aligned}$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} <$ for $i = 0, 1, 2, ..., n-1$. Now

$$\begin{aligned}|Q(z)| \ge& |\alpha_n||z|^{n+1} - \Big\{\big(|a_n - a_{n-1}||z|^n + ... + |a_{m+1} - a_m||z|^{m+1} + |a_m - a_{m-1}||z|^m + ... + |a_1 - a_0||z| + a_0\big)\\ &+ \big(|b_n - b_{n-1}||z|^n + ... + |b_{m+1} - b_m||z|^{m+1} + |b_m - b_{m-1}||z|^m + ... + |b_1 - b_0||z| + b_0\big)\Big\}\\ \ge& |a_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Bigg\{\Bigg(|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + ... + \frac{|a_{m+1} - a_m|}{|z|^{n-m-1}} + \frac{|a_m - a_{m-1}|}{|z|^{n-m}} + ... + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\Bigg)\\ &+ \Bigg(|b_n - b_{n-1}| + \frac{|b_{n-1} - b_{n-2}|}{|z|} + ... + \frac{|b_{m+1} - b_m|}{|z|^{n-m-1}} + \frac{|b_m - b_{m-1}|}{|z|^{n-m}} + ... + \frac{|b_1 - b_0|}{|z|^{n-1}} + \frac{|b_0|}{|z|^n}\Bigg)\Bigg\}\Bigg]\\ \ge& |a_n||z|^n\Bigg[|z| - \frac{1}{|a_n|}\Bigg\{\big(|ka_n - a_{n-1} - ka_n + a_n| + |a_{n-1} - a_{n-2}| + ... + |a_{m+1} - k_1a_m + k_1a_m - a_m|\\ &+ |a_m - k_1a_m + k_1a_m - a_{m-1}| + ... + |a_1 + \delta - a_0 - \delta| + |a_0|\big) + \big(|lb_n - b_{n-1} - lb_n + b_n| + |b_{n-1} - b_{n-2}|\\ &+ ... + |b_{m+1} - l_1b_m + l_1b_m - b_m| + |b_m - l_1b_m + l_1b_m - b_{m-1}| + ... + |b_1 + \eta - b_0 - \eta| + |b_0|\big)\Bigg\}\Bigg]\\ \ge& |a_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Bigg\{\big[(ka_n - a_{n-1}) + (k-1)|a_n| + (a_{n-1} - a_{n-2}) + ... + (a_{m+1} - k_1a_m) + (k_1 - 1)|a_m|\\ &+ (k_1a_m - a_{m-1}) + (k_1 - 1)|a_m| + .. + (a_1 + \delta - a_0) + \delta + |a_0|\big] + \big[(lb_n - b_{n-1}) + (l-1)|b_n| + (b_{n-1} - b_{n-2})\\ &+ ... + (b_{m+1} - l_1b_m) + (l_1 - 1)|b_m| + (l_1b_m - b_{m-1}) + (l_1 - 1)|b_m| + .. + (b_1 + \eta - b_0) + \eta + |b_0|\big]\Bigg\}\Bigg]\\ =& |a_n||z|^n\Bigg[|z| - \frac{1}{|\alpha_n|}\Bigg\{k(a_n + |a_n|) + l(b_n + |b_n|) + |a_0| + |b_0| + 2(k_1 - 1)|a_m| + 2(l_1 - 1)|b_m|\end{aligned}$$

$$-(a_0+b_0+|a_n|+|b_n|)+2\delta+2\eta\Big\}\Big]>0$$

provided

$$|z|>\frac{1}{|\alpha_n|}\Big[k(a_n+|a_n|)+l(b_n+|b_n|)+|a_0|+|b_0|+2(k_1-1)|a_m|+2(l_1-1)|b_m|-(a_0+b_0+|a_n|+|b_n|)+2\delta+2\eta\Big].$$

This shows that $Q(z)>0$ provided

$$|z|>\frac{1}{|\alpha_n|}\Big[k(a_n+|a_n|)+l(b_n+|b_n|)+|a_0|+|b_0|+2(k_1-1)|a_m|+2(l_1-1)|b_m|-(a_0+b_0+|a_n|+|b_n|)+2\delta+2\eta\Big].$$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$$|z|\leq\frac{1}{|\alpha_n|}\Big[\big[k(a_n+|a_n|)+l(b_n+|b_n|)+|a_0|+|b_0|+2(k_1-1)|a_m|+2(l_1-1)|b_m|-(a_0+b_0+|a_n|+|b_n|)+2\delta+2\eta\Big].$$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem 1.1. □

*Proof of Theorem 1.8.* Let $P(z)=\alpha_nz^n+\alpha_{n-1}z^{n-1}+...+\alpha_{m-1}z^{m-1}+\alpha_mz^m+\alpha_{m+1}z^{m+1}+...+\alpha_1z+\alpha_0$ be a polynomial of degree n. Then consider the polynomial $Q(z)=(1-z)P(z)$ so that

$$\begin{aligned}Q(z)=&-\alpha_nz^{n+1}+(\alpha_n-\alpha_{n-1})z^n+...+(\alpha_{m+1}-\alpha_m)z^{m+1}+(\alpha_m-\alpha_{m-1})z^m+...+(\alpha_1-\alpha_0)z+\alpha_0.\\ =&-\alpha_nz^{n+1}+(a_n-a_{n-1})z^n+...+(a_{m+1}-a_m)z^{m+1}+(a_m-a_{m-1})z^m+...+(a_1-a_0)z+a_0+\\ &+i\big\{(b_n-b_{n-1})z^n+...+(b_{m+1}-b_m)z^{m+1}+(b_m-b_{m-1})z^m+...+(b_1-b_0)z+b_0\big\}.\end{aligned}$$

Also if $|z|>1$ then $\frac{1}{|z|^{n-i}}<1$ for $i=0,1,2,...,n-1$. Now

$$\begin{aligned}|Q(z)|\geq&|\alpha_n||z|^{n+1}-\Big\{\big(|a_n-a_{n-1}||z|^n+...+|a_{m+1}-a_m||z|^{m+1}+|a_m-a_{m-1}||z|^m+...+|a_1-a_0||z|+a_0\big)\\ &+\big(|b_n-b_{n-1}||z|^n+...+|b_{m+1}-b_m||z|^{m+1}+|b_m-b_{m-1}||z|^m+...+|b_1-b_0||z|+b_0\big)\Big\}\\ \geq&|a_n||z|^n\Bigg[|z|-\frac{1}{|\alpha_n|}\Bigg\{\Bigg(|a_n-a_{n-1}|+\frac{|a_{n-1}-a_{n-2}|}{|z|}+...+\frac{|a_{m+1}-a_m|}{|z|^{n-m-1}}+\frac{|a_m-a_{m-1}|}{|z|^{n-m}}+...+\frac{|a_1-a_0|}{|z|^{n-1}}+\frac{|a_0|}{|z|^n}\Bigg)\\ &+\Bigg(|b_n-b_{n-1}|+\frac{|b_{n-1}-b_{n-2}|}{|z|}+...+\frac{|b_{m+1}-b_m|}{|z|^{n-m-1}}+\frac{|b_m-b_{m-1}|}{|z|^{n-m}}+...+\frac{|b_1-b_0|}{|z|^{n-1}}+\frac{|b_0|}{|z|^n}\Bigg)\Bigg\}\Bigg]\\ \geq&|a_n||z|^n\Bigg[|z|-\frac{1}{|a_n|}\Big\{\big(|ra_n-a_{n-1}-ra_n+a_n|+|a_{n-1}-a_{n-2}|+...+|a_{m+1}-sa_m+sa_m-a_m|\\ &+|a_m-sa_m+sa_m-a_{m-1}|+...+|a_1+\delta-a_0-\delta|+|a_0|\big)\big(|xb_n-b_{n-1}-xb_n+b_n|+|b_{n-1}-b_{n-2}|\\ &+...+|b_{m+1}-yb_m+yb_m-b_m|+|b_m-yb_m+yb_m-b_{m-1}|+...+|b_1+\eta-b_0-\eta|+|b_0|\big)\Big\}\Bigg]\\ \geq&|a_n||z|^n\Bigg[|z|-\frac{1}{|\alpha_n|}\Big\{\big[(a_{n-1}-ra_n)+(1-r)|a_n|+(a_{n-2}-a_{n-1})+...+(sa_m-a_{m+1})+(1-s)|a_m|\\ &+(a_{m-1}-sa_m)+(1-s)|a_m|+..+(a_0+\delta-a_1)+\delta+|a_0|\big]+\big[(b_{n-1}-xb_n)+(1-x)|b_n|+(b_{n-2}-b_{n-1})\\ &+...+(yb_m-b_{m+1})+(1-y)|b_m|+(b_{m-1}-yb_m)+(1-y)|b_m|+..+(b_0+\eta-b_1)+\eta+|b_0|\big]\Big\}\Bigg]\\ =&|a_n||z|^n\Bigg[|z|-\frac{1}{|\alpha_n|}\Big\{a_0+b_0+|a_0|+|b_0|+|a_n|+|b_n|-r(a_n+|a_n|)-x(b_n+|b_n|)\\ &+2\big[|a_m|+|b_m|-s|a_m|-y|b_m|+\delta+\eta\big]\Big\}\Bigg]>0\end{aligned}$$

provided

$$|z|>\frac{1}{|\alpha_n|}\Big[a_0+b_0+|a_0|+|b_0|+|a_n|+|b_n|-r(a_n+|a_n|)-x(b_n+|b_n|)+2\big[|a_m|+|b_m|-s|a_m|-y|b_m|+\delta+\eta\big]\Big].$$

This shows that $Q(z)>0$ provided

$$|z|>\frac{1}{|\alpha_n|}\Big[a_0+b_0+|a_0|+|b_0|+|a_n|+|b_n|-r(a_n+|a_n|)-x(b_n+|b_n|)+2\big[|a_m|+|b_m|-s|a_m|-y|b_m|+\delta+\eta\big]\Big].$$

Hence all the zeros of $Q(z)$ with $|z|>1$ lie in

$$|z|\leq\frac{1}{|\alpha_n|}\Big[a_0+b_0+|a_0|+|b_0|+|a_n|+|b_n|-r(a_n+|a_n|)-x(b_n+|b_n|)+2\big[|a_m|+|b_m|-s|a_m|-y|b_m|+\delta+\eta\big]\Big].$$

But those zeros of $Q(z)$ whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of $P(z)$ are also the zeros of $Q(z)$ lie in the circle defined by the above inequality and this completes the proof of the Theorem 1.8. □

## References


[1] A. Joyal, G. Labelle and Q. I. Rahman, *On the location of zeros of polynomials*, Canad. Math. Bull., 10(1967), 53-63.

[2] A. Aziz and B. A. Zargar, *Some extensions of Eneström-Kakeya Theorem*, Glasnik Matematiki, 31(1996), 239-244.

[3] G. Eneström, *Remarquee sur un théorème relatif aux racines de l'equation* $a_n+...+a_0=0$ *oü tous les coefficient sont et positifs*, Tôhoku Math. J., 18(1920), 34-36.

[4] Gulshan Singh, *On the Eneström-Kakeya Theorem*, American Journal of Mathematical Analysis, 2(1)(2014), 15-18.

[5] P. Ramulu, *Some Generalization of Enestrm-Kakeya Theorem*, International Journal of Mathematics and Statistics Invention, 3(2)(2015), 52-59.

[6] P. Ramulu, G. L. Reddy and C.Gangadhar, *On the zeros of polynomials with real coefficients*, International Journal of Applied Mathematical Sciences, 5(2)(2015), 109-118.

[7] P. Ramulu and G.L. Reddy, *On the Enestrom Kakeya theorem*, International Journal of Pure and Applied Mathematics, 102(4)(2015), 687-700.

[8] R. Fricke, *Lehrbuch der Algebra*, Friedrich Vieweg and Sohn Verlag, braunschweig, Germany, (1924).

[9] S. Kakeya, *On the limits of the roots of an alegebraic equation with positive coefficient*, Tôhoku Math. J., 2(1912-1913), 140-142.

[10] W. Heitzinger, W. I. Troch and G. Valentin, *Praxis nichtlinearer Gleichungen*, Carl Hanser Verlag, munchen, Wien, Germany, Austria, (1985).

[11] W. M. Shah, A. Liman and Shamim Ahmad Bhat, *On the Enestrm-Kakeya Theorem*, International Journal of Mathematical Science, 7(1-2)(2008), 111-120.

[12] W. M. Shah and A. Liman, *On the zeros of a certain class of polynomials and related analytic functions*, Mathematicka Balkanicka, New Series, 19(2005), 3-4.